Class 23, given on Feb 24, 2010, for Math 13, Winter 2010

1. Examples of direction calculation of surface integrals

Let's look at a few more examples of how one can directly calculate surface integrals.

Examples.

- (Exercise #21, Chapter 17.7 of textbook) Let $\mathbf{F} = \langle xze^y, -xze^y, z\rangle$, and let S be the part of the plane $x + y + z = 1$ in the first octant with downward orientation. Find the flux of $\mathbf{F}$ across S.

  This time we cannot calculate $\mathbf{F}\cdot\mathbf{n}$ as quickly as the previous two examples, but we still are in a situation where our calculations are not as complicated as they otherwise might be. First, we find a parameterization for S; since S is the graph of the function $z = 1 - x - y$ over the domain $x \geq 0, y \geq 0, x + y \leq 1$, we can use

  $$\mathbf{r}(u, v) = \langle u, v, 1 - u - v\rangle, u + v \leq 1, 0 \leq u, v.$$

  We calculate $\mathbf{r}_u \times \mathbf{r}_v$ for this choice of $\mathbf{r}$. Since $\mathbf{r}_u = \langle 1, 0, -1\rangle, \mathbf{r}_v = \langle 0, 1, -1\rangle$, $\mathbf{r}_u \times \mathbf{r}_v = \langle 1, 1, 1\rangle$. Notice that this points in the wrong direction, so we should remember to reverse the sign of our answer at the end of our calculations. In any case, for this choice of $\mathbf{r}_u \times \mathbf{r}_v$,

  $$\mathbf{F}\cdot\mathbf{r}_u \times \mathbf{r}_v = \langle xze^y, -xze^y, z\rangle \cdot \langle 1, 1, 1\rangle = z.$$

  As a function of u, v, this is equal to $1 - u - v$.

  To find the surface integral we are interested in, we need to calculate the double integral of this function over the domain D of the uv plane which describes S, which is $u, v \geq 0, u + v \leq 1$. This can be described using inequalities $0 \leq u \leq 1, 0 \leq v \leq 1 - u$. Therefore, the integral we want to calculate is equal to

  $$\iint_D (1 - u - v)\, dA = \int_0^1 \int_0^{1-u} 1 - u - v\, dv\, du = \int_0^1 \frac{(1-u)^2}{2}\, du = \frac{-(1-u)^3}{6}\Big|_0^1 = \frac{1}{6}.$$

  Recall that we need to reverse the sign, since our choice of $\mathbf{r}$ yielded the incorrect direction of $\mathbf{r}_u \times \mathbf{r}_v$, so the answer is $-1/6$.

  In this example, because we could not rapidly calculate $\mathbf{F}\cdot\mathbf{n}$, we carried out all the steps that are usually needed to evaluate a surface integral. Nevertheless, in this example $\mathbf{r}_u \times \mathbf{r}_v$ was constant, so our resulting calculations were relatively simple. Notice that if we choose to just integrate $\mathbf{F}\cdot\mathbf{r}_u \times \mathbf{r}_v$, we can skip the calculation of $\mathbf{n}$.
- Let $\mathbf{F} = \langle y, x, z\rangle$, and let S be the cylinder $x^2 + y^2 = 1, -1 \leq z \leq 1$, with outward orientation. (S is not closed, but by outward we mean the outward orientation on S if we had included the top and bottom of the cylinder.)

  We'll evaluate the surface integral of $\mathbf{F}$ through S directly. First, we need to find a parameterization for S. Because S is a cylinder of radius 1, centered at the origin, it makes sense to use cylindrical coordinates. If we let $X(u, v) = \cos u, Y(u, v) = \sin u, Z(u, v) = v$, then $\mathbf{r}(u, v) = \langle \cos u, \sin u, v\rangle, 0 \leq u \leq 2\pi, -1 \leq v \leq 1$, is a parameterization of S. Let D be the region $0 \leq u \leq 2\pi, -1 \leq v \leq 1$ in the uv plane. We then calculate $\mathbf{r}_u \times \mathbf{r}_v$:

  $$\mathbf{r}_u = \langle -\sin u, \cos u, 0\rangle, \mathbf{r}_v = \langle 0, 0, 1\rangle, \mathbf{r}_u \times \mathbf{r}_v = \langle \cos u, \sin u, 0\rangle.$$

We check that $\mathbf{r}_u \times \mathbf{r}_v$ points in the same direction as the specified orientation. One can quickly do this by sketching a few values of $\mathbf{r}_u \times \mathbf{r}_v$ at various points of S. To evaluate the surface integral of $\mathbf{F}$ across S, we compute the double integral

$$\iint\limits_D \mathbf{F} \cdot \mathbf{r}_u \times \mathbf{r}_v \, dA = \int_{-1}^{1} \int_0^{2\pi} \langle Y(u,v), X(u,v), Z(u,v) \rangle \cdot \langle \cos u, \sin u, 0 \rangle \, du \, dv$$

$$= \int_{-1}^{1} \int_0^{2\pi} 2 \sin u \cos u \, du \, dv = \int_{-1}^{1} \sin^2 u \Big|_{u=0}^{u=2\pi} dv = \int_{-1}^{1} 0 \, dv = 0.$$

2. The Divergence Theorem

We now want to discuss one of the great theorems of this subject. It has great practical utility in electromagnetism, but is also of substantial theoretical interest. We will call this theorem the Divergence Theorem, but it goes under several different names, including Gauss' Theorem and Ostrogradsky's Theorem.

Let E be some sort of solid connected region in $\mathbb{R}^3$ with piecewise smooth boundary, and let S be its boundary with default outward orientation. For example, E might be a solid cube, a solid sphere, a solid cylinder, a solid hemisphere, etc. Let $\mathbf{F}$ be a C^1 vector field defined on E. Then the Divergence Theorem says:

Theorem. (The Divergence Theorem) With setup as above,

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint\limits_E \nabla \cdot \mathbf{F} \, dV.$$

We can see why this is called the divergence theorem, since the right hand integral contains the divergence of a vector field in its integrand. Notice the philosophical similarity of this theorem to Green's Theorem: a surface integral of a vector field is equal to the triple integral of some related scalar function on the interior of the surface. Green's Theorem tells us that the line integral of a vector field over a closed curve is equal to the double integral of some related scalar function over the interior of the closed curve. The actual function which appears in the double or triple integral depends on $\mathbf{F}$ in a different way in each of these theorems, but the general idea is the same.

In terms of calculations, the divergence theorem is frequently used to simplify the calculation of certain surface integrals, if one can easily integrate $\nabla \cdot \mathbf{F}$ on the corresponding three dimensional region. This is especially true if E is a polyhedron, like a cube or rectangular prism, since it is usually easy to evaluate triple integrals over these regions. Notice the similarity between this idea and the idea that using Green's Theorem is usually a good technique when integrating over rectangular paths.

Examples.

- Let S be the cube with vertices $(\pm 1, \pm 1, \pm 1)$ with outward orientation and let $\mathbf{F}(x, y, z) = \langle y^2 + x, 2x + \sin z, z^2 \rangle$. Evaluate the flux of $\mathbf{F}$ across S.

  As you might imagine, trying to directly calculate this surface integral would be very difficult, since $\mathbf{F}$ does not behave simply on any of the faces of S, and S has 6 faces, which means you would have to calculate 6 different integrals. Instead, let's use the divergence theorem.

  The solid E is the solid bounded by S, which is $-1 \leq x, y, z \leq 1$. We also have $\nabla \cdot \mathbf{F} = 1 + 0 + 2z$. Therefore,

$$\iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint\limits_E 1+2z\,dV = \int_{-1}^1\int_{-1}^1\int_{-1}^1 1+2z\,dz\,dy\,dx = 4\int_{-1}^1 1+2z\,dz = 4(z+z^2)\Big|_{-1}^1 = 8.$$

This is much simpler than calculating six different, difficult surface integrals!

- (A question from Peter de Boursac) Suppose S is a closed surface of any shape, and suppose $\mathbf{F}$ is a constant vector field. Then $\nabla \cdot \mathbf{F} = 0$, so

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\,dS = \iiint\limits_E 0\,dV = 0.$$

More generally, the flux across any closed surface of a divergence-free vector field (sometimes called incompressible) will be 0.

The divergence theorem also makes precise the idea that divergence of a vector field measures the tendency of the vector field to measure the rate at which something is leaving or entering a point. More precisely, suppose we are interested in the value of $\nabla\cdot\mathbf{F}$ at a point P. Draw a small box E, of dimensions $\Delta x, \Delta y, \Delta z$ around P, and let S be the boundary of E with outward orientation. Then the flux of $\mathbf{F}$ across S is a measure of the amount of the vector field which is flowing into (if flux has negative sign) or out of (if flux has positive sign) E. On the other hand,

$$\iint\limits_S \mathbf{F} \cdot \mathbf{n}\,dS = \iiint\limits_E \nabla \cdot \mathbf{F}\,dV.$$

As $\Delta x, \Delta y, \Delta z$ become very small, $\nabla\cdot\mathbf{F}$ will be approximately constant, so we can estimate the left hand side relatively accurately with

$$\nabla \cdot \mathbf{F}(P) \iiint\limits_E 1\,dV = \nabla \cdot \mathbf{F}(P)\Delta x\Delta y\Delta z.$$

Therefore, if we divide both sides of the second to last equation by $\Delta x\Delta y\Delta z = V(E)$, we have

$$\frac{1}{V(E)} \iint\limits_S \mathbf{F} \cdot \mathbf{n}\,dS \approx \nabla \cdot \mathbf{F}(P).$$

The left hand side can be interpreted as the rate at which the vector field flows out of E per unit volume, so this mathematically shows why one can interpret $\nabla \cdot \mathbf{P}$ as describing the rate at which a vector field flows into or out of a point.

Even though the Divergence Theorem is only stated for a closed surface S, you can sometimes apply it to the calculation of flux through non-closed surfaces by adding pieces of surfaces to S to make it closed.

Examples.

- Consider the cylindrical example we saw earlier today: let $\mathbf{F} = \langle y, x, z\rangle$, and let S be the cylinder $x^2 + y^2 = 1, -1 \le z \le 1$, with outward orientation. We'll use the Divergence Theorem to calculate this surface integral in a different way.

  We make S into a closed surface S' by including its top and bottom, which correspond to $z = \pm 1, x^2 + y^2 \le 1$. Let E be the cylinder which S' encloses. We give them the appropriate orientation to make S have outward pointing orientation.

  Since $\nabla \cdot \mathbf{F} = 0 + 0 + 1$, the Divergence Theorem tells us

$$\iint\limits_{S'} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint\limits_{E} 1\,dV = V(E).$$

Since E is a cylinder of height 2 and base $\pi 1^2 = \pi$, the volume of E is 2π, so the above expression equals 2π. On the other hand, the flux of $\mathbf{F}$ across S' is the sum of the flux across S, the top of S', and the bottom of S'. We can evaluate the flux across the top and bottom of S' very easily. For example, for the top of S', $\mathbf{n} = \langle 0,0,1\rangle$, so $\mathbf{F}\cdot\mathbf{n} = \langle y,x,1\rangle \cdot 001 = 1$. Therefore, the flux across the top is 1 times the area of the top, which is just π. Similarly, one can show that the flux across the bottom is also π. Therefore,

$$2\pi = \iint\limits_{S'} \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_{S} \mathbf{F}\cdot\mathbf{n}\,dS + \iint\limits_{\text{top}} \mathbf{F}\cdot\mathbf{n}\,dS + \iint\limits_{\text{bottom}} \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_{S} \mathbf{F}\cdot\mathbf{n}\,dS + \pi + \pi \Rightarrow \iint\limits_{S} \mathbf{F}\cdot\mathbf{n}\,dS = 0.$$

- The idea behind the above example is that we can replace the calculation of a surface integral across a non-closed surface with the calculation of another surface integral over a different non-closed surface, which together with the original surface form a closed surface which encases a solid E, and then also calculate the integral of $\nabla\cdot\mathbf{F}$ over E. While this seems like we're replacing one problem with another which involves more work (after all, we need to calculate an additional triple integral), this sometimes turns out to involve less calculation if the original surface is very complicated, but the secondary surface and $\nabla\cdot\mathbf{F}$ turn out to be simple.

  For example, let S be the surface given by the the part of the sphere $x^2+y^2+z^2 = 4$ with $z \leq 1$, with outward pointing orientation, and let $\mathbf{F} = \langle e^z, xz, x^2+y^2\rangle$. If we wanted to directly calculate the surface integral of $\mathbf{F}$ across S, we would need to parameterize S using spherical coordinates (not pleasant), and the expression for $\mathbf{F}\cdot\mathbf{n}$ would be very complicated.

  Instead, we will carry out the technique described above. Let's complete S to a closed surface by adding a cap S' which consists of points $x^2+y^2 \leq 3, z = 1$. (You have flexibility in choosing how you want to complete S, but you want to make the choice which will result in the simplest possible calculations. Determining the proper choice is something you gain with experience.) To ensure that S and S' have compatible orientations, we need S' to have orientation pointing upwards.

  Furthermore, we quickly see that $\nabla\cdot\mathbf{F} = 0$. If we let E be the solid that S, S' enclose, we have

$$\iint\limits_{S} \mathbf{F}\cdot\mathbf{n}\,dS + \iint\limits_{S'} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint\limits_{E} \nabla\cdot\mathbf{F}\,dV = \iiint\limits_{E} 0\,dV = 0.$$

Therefore,

$$\iint\limits_{S} \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_{S'} -\mathbf{F}\cdot\mathbf{n}\,dS.$$

Although the left hand side is difficult to directly calculate, the right hand side is easy to calculate. Since $\mathbf{n} = \langle 0,0,1\rangle$ on S' (since S' is parallel to the xy plane, and has upwards orientation), we have $\mathbf{F}\cdot\mathbf{n}\,dS = x^2+y^2$ on S'. Furthermore, S' is easily parameterized by $\mathbf{r}(u,v) = \langle u,v,1\rangle, u^2+v^2 \leq 3$. Therefore,

$$\iint_{S'} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{D} u^2 + v^2 \, dA$$

where D is the domain $u^2 + v^2 \leq 3$. This looks like an integral we should use polar coordinates to solve; letting $u = r\cos\theta, v = r\sin\theta$, we have

$$\iint_{D} u^2 + v^2 \, dA = \int_0^{2\pi} \int_0^{\sqrt{3}} r^2 r \, dr \, d\theta = \int_0^{2\pi} \frac{r^4}{4}\Big|_{r=0}^{r=\sqrt{3}} d\theta = 2\pi \frac{9}{4} = \frac{9\pi}{2}.$$

Therefore, the original integral we wanted to calculate (which was over S instead of S') is equal to $-9\pi/2$.